Year 12 Semester 2 Examination, 2017

Question/Answer Booklet

Hale School

# MATHEMATICS SPECIALIST

## Section Two
## Calculator Assumed

Student Name

Teacher: (circle) Mr Hill Mr Bausor

Score: (out of 98)

## Time allowed for this section
Reading time before commencing work: ten minutes
Working time for this section: one hundred minutes

## Materials required/recommended for this section
***To be provided by the supervisor***
This Question/Answer Booklet
Formula Sheet (retained from Section One)

***To be provided by the candidate***
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in the WACE examinations

## Important note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

## Structure of this paper

| Section | Number of questions available | Number of questions to be answered | Working time (minutes) | Marks available | Percentage of exam |
|---|---|---|---|---|---|
| Section One: Calculator-free | 7 | 7 | 50 | 51 | 35 |
| Section Two: Calculator-assumed | 12 | 12 | 100 | 98 | 65 |
| | | | **Total** | 149 | 100 |

## Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

**Section Two: Calculator-assumed** **(98 Marks)**

This section has **twelve (12)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

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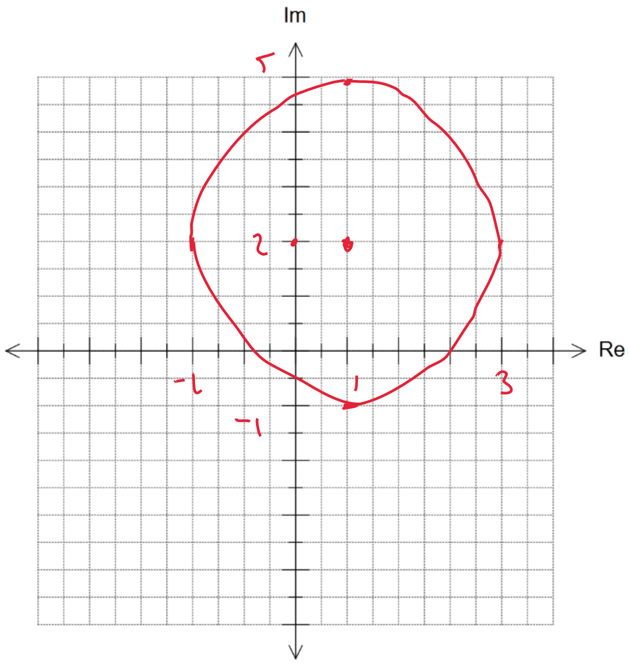
**Question 8** **(6 marks)**

Sketch the following loci on the complex planes provided:

(a) $|2i+1-z|=3$ (3 marks)

$|z-(2i+1)|=3$



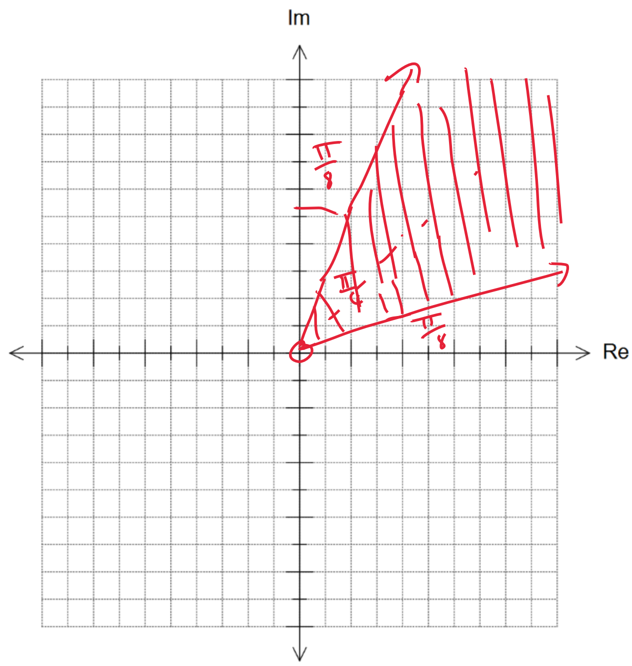

(b) $\dfrac{\pi}{4} \le \arg(z^2) \le \dfrac{3\pi}{4}$ (3 marks)

$\frac{\pi}{4} \le \arg(z^2) \le \frac{3\pi}{4}$

$\frac{\pi}{4} \le \arg(\text{cis}\, 2\theta) \le \frac{3\pi}{4}$

$\frac{\pi}{4} \le 2\theta \le \frac{3\pi}{4}$

$\frac{\pi}{8} \le \theta \le \frac{3\pi}{8}$



**Question 9** **(7 marks)**

Three position vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are defined as follows:

$$\underset{\sim}{a} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}, \ \underset{\sim}{b} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}, \ \underset{\sim}{c} = \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}.$$

(a) Determine the vector in the same direction as $\underset{\sim}{a}$ and equal in magnitude to $\underset{\sim}{c}$. (2 marks)

$$\hat{\underset{\sim}{a}} = \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \qquad |\underset{\sim}{c}| = 5$$

$$\therefore \text{vector} = \frac{5}{\sqrt{14}} \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$$

(b) Determine the equation of the plane $\Pi$ containing $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$. (3 marks)

$$\vec{AB} = -\underset{\sim}{a} + \underset{\sim}{b} = \begin{pmatrix} 0 \\ -2 \\ 7 \end{pmatrix} \quad \vec{AC} = -\underset{\sim}{a} + \underset{\sim}{c} = \begin{pmatrix} -4 \\ -3 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 2 \\ -7 \end{pmatrix} \times \begin{pmatrix} 4 \\ 3 \\ -6 \end{pmatrix} = \begin{pmatrix} 9 \\ -28 \\ -8 \end{pmatrix}$$

$$\underset{\sim}{r} \cdot \begin{pmatrix} 9 \\ -28 \\ -8 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 9 \\ -28 \\ -8 \end{pmatrix}$$

$$\underset{\sim}{r} \cdot \begin{pmatrix} 9 \\ -28 \\ -8 \end{pmatrix} = -59$$

(c) Show that the line $\underset{\sim}{r} = \begin{pmatrix} -3 \\ 2+2\lambda \\ -3-7\lambda \end{pmatrix}$ is contained within $\Pi$. (2 marks)

$$\begin{pmatrix} -3 \\ 2+2\lambda \\ -3-7\lambda \end{pmatrix} \cdot \begin{pmatrix} 9 \\ -28 \\ -8 \end{pmatrix} = -27 - 56 - 56\lambda + 24 + 56\lambda$$

$$= -59$$

$\therefore$ contained within $\Pi$.

**Question 10** **(6 marks)**

The mean mass of a chocolate brownie is 18.5 $g$ with a standard deviation of 2.3 $g$.

(a) Sean baked 32 brownies for his Friday Mathematics class festivities. Estimate the probability that this sample of brownies has a mass not exceeding 610 $g$. (2 marks)

$\bar{x} = 18.5 \quad \sigma = \frac{2.3}{\sqrt{32}} = 0.4066$

$\bar{X} \sim N(18.5,\ (0.4066)^2)$

$P(\bar{x} < \frac{610}{32}) = 0.9167$

(b) 80% of samples of 32 brownies each have a mean mass above $x$ $g$. Find $x$. (2 marks)

$P(\bar{x} > k) = 0.8$

$k = 18.1578$

$\times 32$

$= 581.05$ g

(c) The number of brownies, $n$, required for an end of year Mathematics celebration is to meet the following condition; the size $n$ of the random sample of brownies must yield a sampling distribution with a standard deviation of less than 0.2 g. Determine the size $n$ of the sample required. (2 marks)

$s = \frac{\sigma}{\sqrt{n}}$

$0.2 = \frac{2.3}{\sqrt{n}}$

$n = 132.25$

$\therefore n = 133$ brownies.

**Question 11** **(11 marks)**

A child seated on a mat slides down a spiral-shaped water slide. At time t seconds after starting to slide, the position vector of the centre of the mat relative to an origin O at ground level is given by: $\underset{\sim}{r}(t) = 5\sin\left(\frac{\pi}{6}t\right)\underset{\sim}{i} + 5\cos\left(\frac{\pi}{6}t\right)\underset{\sim}{j} + \left(24.5 - \frac{t^2}{8}\right)\underset{\sim}{k}$,

where $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are perpendicular horizontal unit vectors and $\underset{\sim}{k}$ is a unit vector in the vertical direction. Displacement components are measured in metres.

(a) Find the height, in metres, of the start of the water slide above the ground. (1 mark)

$24.5$

(b) Show that the time taken to slide down from the top to the bottom of the slide, which is at ground level, is 14 seconds. (1 mark)

$24.5 - \frac{t^2}{8} = 0$

$t^2 = 196$ $\quad t = 14$

(c) Find the time taken for the child to complete the first loop of the spiral that is vertically below the starting point. (1 mark)

$T = \frac{2\pi}{n} = 12$ secs

(d) Find the speed of the child when ground level is reached. (2 marks)

$\dot{r}(t) = \frac{5\pi}{6}\cos\left(\frac{\pi}{6}t\right)\underset{\sim}{i} - \frac{5\pi}{6}\sin\left(\frac{\pi}{6}t\right)\underset{\sim}{j} + \left(-\frac{t}{4}\right)\underset{\sim}{k}$

$|\dot{r}(14)| = \sqrt{\left(\frac{5\pi}{6}\right)^2(\cos^2 + \sin^2) + \left(\frac{14}{4}\right)^2}$

$= 4.37 \ \sim 1\,\mathrm{s}$

***Question 11 continued…***

(e) Show that the magnitude of the child's acceleration is constant. (3 marks)

$\ddot{r}(t) = 5\left(\frac{\pi}{6}\right)^2 \sin\left(\frac{\pi t}{6}\right)\underset{\sim}{i} - 5\left(\frac{\pi}{6}\right)^2 \cos\left(\frac{\pi t}{6}\right)\underset{\sim}{j} - \frac{1}{4}\underset{\sim}{k}$

$|\ddot{r}(t)| = \sqrt{5^2\left(\frac{\pi}{6}\right)^4\left(\sin^2\left(\frac{\pi t}{6}\right) + \cos^2\left(\frac{\pi t}{6}\right)\right) + \left(\frac{1}{4}\right)^2}$

$= \sqrt{5^2\left(\frac{\pi}{8}\right)^4 + \left(\frac{1}{4}\right)^2}$ (const)

(f) Write down a definite integral in the form $\int_{t_0}^{t_1} \sqrt{a+bt^2}\, dt$ which represents the distance travelled by the child from the start to the finish of the slide. Evaluate the distance travelled by the child, correct to the nearest tenth of a metre. (3 marks)

$\int_0^{14} |v(t)|\, dt$

$= \int_0^{14} \sqrt{\left(\frac{5\pi}{6}\right)^2 + \frac{1}{16}t^2}\, dt$

$= 45.7$ m

**Question 12** **(11 marks)**

Let $z_1 = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$ .

(a) Show that $z_1$ is a root of the equation $z^3 - 1 = 0$ . (3 marks)

$\left(\text{cis}\left(\frac{2\pi}{3}\right)\right)^3 - 1$

$= (\text{cis}\, 2\pi) - 1$

$= 1 - 1$

$= 0$

(b) Determine the other two roots of the equation $z^3 - 1 = 0$ . (2 marks)

$z_2 = \cos\left(\frac{2\pi}{3}\right) - i\sin\left(\frac{2\pi}{3}\right)$

$z_3 = 1$

(c) Use the roots found in (b) roots to show that $z^3 - 1 = (z-1)(z^2 + z + 1)$ (2 marks)

$(z-1)\left(z - \text{cis}\,\frac{2\pi}{3}\right)\left(z - \text{cis}\,\frac{-2\pi}{3}\right)$

$= (z-1)\left(z^2 - \text{cis}\,\frac{2\pi}{3}\, z - \text{cis}\left(\frac{-2\pi}{3}\right) z + \text{cis}\,\frac{2\pi}{3}\, \text{cis}\,\frac{-2\pi}{3}\right)$

$= (z-1)\left(z^2 - 2\cos\frac{2\pi}{3}\, z + 1\right)$

$= (z-1)\left(z^2 + 2\cdot\frac{1}{2} z + 1\right)$

$= (z-1)(z^2 + z + 1)$

***Question 12 continued...***

(c) Deduce that $z_1^{\ 2} + z_1 + 1 = 0$ and use this result to show that $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$ .

(4 marks)

if $(z_1 - 1)(z_1^2 + z_1 + 1) = 0$

$\Rightarrow z_1^2 + z_1 + 1 = 0$

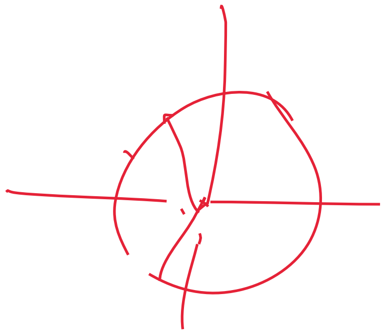
$\left(\text{cis}\left(\frac{2\pi}{3}\right)\right)^2 + \text{cis}\frac{2\pi}{3} + 1 = 0$

$\text{cis}\frac{4\pi}{3} + \text{cis}\frac{2\pi}{3} + 1 = 0$

Re: $\cos\frac{4\pi}{3} + \cos\frac{2\pi}{3} + 1 = 0$

$2\cos\frac{2\pi}{3} + 1 = 0$

$\cos\frac{2\pi}{3} = -\frac{1}{2}$



**Question 13** **(10 marks)**

A random sample of 30 households in Innaloo were selected as part of a study on electricity consumption. The mean spring consumption was 18.7 kWh of electricity per day. In a very large study the previous year, it was found that the standard deviation of spring household electricity consumption was 3.2 kWh per day.

(a) Calculate a 95% confidence interval for the mean daily electricity consumption of households in Innaloo. (2 marks)

[17.555, 19.845]

$$18.7 \pm 1.95996\frac{3.2}{\sqrt{30}} = [\cancel{4.988}, 5.\cancel{612}]$$

| |
|---|
| Calculates z score |
| Correct Interval |

(b) The Premier of Western Australia, Mark McGowan, claimed that mean spring electricity consumption of households in Perth was 20 kWh per day. Based on this sample, what confidence interval % is required for this claim to be accurate? (3 marks)

$$18.7 + Z\frac{3.2}{\sqrt{30}} = 20$$

$$Z = 2.2251$$

$$X \sim N(0,1)$$

$$P(-2.2251 < X < 2.2251) = 0.9736$$

Require a 97.4% confidence interval.

(c) 65 similar studies are planned for Perth.

(i) Determine the least number of households that should be sampled in each of these studies to be 98% confident that the mean spring electricity of households in Perth is within 0.5 kWh of the true value. (3 marks)

$$n = \left(\frac{2.3263 \times 3.2}{0.5}\right)^2$$

$$= 221.67$$

222 households.

| |
|---|
| Calculates z score |
| Correct formula |
| Rounds up |

(ii) How many of the 98% confidence intervals from these additional studies are expected to contain the true mean? Justify your answer. (2 marks)

$0.98 \times 65$

$= 63.7$

64 of the studies, as we expect 95% of the intervals to contain the true mean.

**Question 14** **(4 marks)**

A mass has acceleration $a \ m/s^2$ given by $a = v^2 - 3$, where $v \ m/s$ is the velocity of the mass when it has a displacement of $x \ m$ from the origin,
Find $v$ in terms of $x$ given that $v = -2 \ m/s$ where $x = 1 \ m$.

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

$$\therefore \quad v\frac{dv}{dx} = v^2 - 3$$

$$\int \frac{v}{v^2-3}\,dv = \int 1\,dx$$

$$\frac{1}{2}\ln|v^2-3| = x + c \qquad (v=-2 \quad x=1)$$

$$\frac{1}{2}\ln(4-3) = 1 + c$$

$$c = -1$$

$$\therefore \quad \frac{1}{2}\ln|v^2-3| = x - 1$$

$$v^2 - 3 = e^{2x-2}$$

$$v^2 = e^{2x-2} + 3$$

$$v = \pm\sqrt{e^{2x-2}+3}$$

**Question 15** **(11 marks)**

A particle is moving along a straight line so that its displacement from a fixed point, O , at time t is given by $x = a\sin nt + b\cos nt$ where $a$ , $b$ and $n$ are constants, where $n > 0$ .

(a) Show that the particle is in simple harmonic motion. (3 marks)

$$\frac{dx}{dt} = an\cos nt - bn\sin nt$$

$$\frac{d^2x}{dt^2} = -an^2\sin nt - bn^2\cos nt$$

$$\frac{d^2x}{dt^2} = -n^2(x)$$

$\therefore$ SHM

(b) Given that initially the particle is 30 cm to the right of O, has a velocity of $-8\ cm/s$ and has an acceleration of $-1.2\ cm/s^2$, determine the values of $a$ , $b$ and $n$.

(4 marks)

$x(0) = 30 \quad v(0) = -8 \quad a(0) = -1.2$

$30 = a\sin 0 + b\cos 0$

$\underline{b = 30}$

$a = -n^2x$

$-1.2 = -n^2(30)$

$\underline{n = 0.2}$

$v = an\cos(0)$

$-8 = a(0.2)$

$\underline{a = 40}$

***Question 15 continued…***

(c) Find all possible positions of the particle when its speed is $2\sqrt{21}\ cm/s$. (4 marks)

$V = 8\cos 0.2t - 6\sin 0.2t$

$2\sqrt{21} = 8\cos 0.2t - 6\sin 0.2t$

$t_1 = -10\tan^{-1}(\sqrt{21} - 4), -10\tan^{-1}\left(\frac{\sqrt{21} - 4}{5}\right)$

$x(t_1) = 20, -20$ . m .

**Question 16** **(10 marks)**

(a) If $cis\theta = \cos\theta + i\sin\theta$ use De Moivre's Theorem to find an expression for $\sin 4\theta$ in terms of $\sin\theta$ and $\cos\theta$. Hence show that $\dfrac{\sin 4\theta}{\sin\theta} = 8\cos^3\theta - 4\cos\theta$. (6 marks)

$\text{Im}[(cis\theta)^4 = (\cos\theta + i\sin\theta)^4]$

$\text{Im}\{cis4\theta = \cos^4\theta + 4\cos^3\theta\sin\theta i - 6\cos^2\theta\sin^2\theta - 4\cos\theta\sin^3\theta i + \sin^4\theta]$

$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

$\dfrac{\sin 4\theta}{\sin\theta} = 4\cos^3\theta - 4\cos\theta\sin^2\theta$

$= 4\cos^3\theta - 4\cos\theta(1-\cos^2\theta)$

$= 8\cos^3\theta - 4\cos\theta$

(b) Hence, determine algebraically $\displaystyle\int \frac{\sin 4\theta}{\sin\theta}\, d\theta$ . (4 marks)

$\displaystyle\int \frac{\sin 4\theta}{\sin\theta}\, d\theta$

$= \displaystyle\int 8\cos^3\theta - 4\cos\theta\ d\theta$

$= \displaystyle\int 8\cos\theta(1-\sin^2\theta) - 4\cos\theta\ d\theta$

$= \displaystyle\int 8\cos\theta - 8\cos\theta\sin^2\theta - 4\cos\theta\ d\theta$

$= +8\sin\theta - \frac{8}{3}\sin^3\theta - 4\sin\theta + c$

**Question 17** **(8 marks)**

An internet marketing company has found that important news spreads through the Year 12 student population according to the formula: $\frac{dN}{dt} = k(P - N)$ where $N$ is the number of people who know the important news $t$ hours after the important news is announced. $P$ and $k$ are constants where $P$ is the total population of Year 12 students and $k$ is the growth factor.

(a) Let $A_0$ be the initial number of people not knowing the news. Use integration to show that $N = P - A_0 e^{-kt}$ is the solution of the differential equation. (4 marks)

$$\frac{dN}{dt} = k(P-N)$$

$$\int \frac{1}{P-N}\, dN = \int k\, dt \quad ✓$$

$$-\ln(P-N) = kt + c \quad ✓$$

$$P - N = e^{-kt-c} \quad ✓$$

$$N = P - e^{-c}e^{-kt}$$

when $t=0$ $N=0$ $\therefore e^{-c} = A_0 = P$

$$\therefore N = P - A_0 e^{-kt} \quad ✓$$

(b) Western Australia has a Year 12 student population of 25 000 students. News that one of three large Drones had malfunctioned in the popular television show "Game of Drones" was released at 9AM. Three hours later, 60% of the student population knew about the news. At what time, to the nearest hour, will 95% of the population know the news? (4 marks)

$$N = 25000 - 25000e^{-kt}$$

$$15\,000 = 25000 - 25000e^{-3k}$$

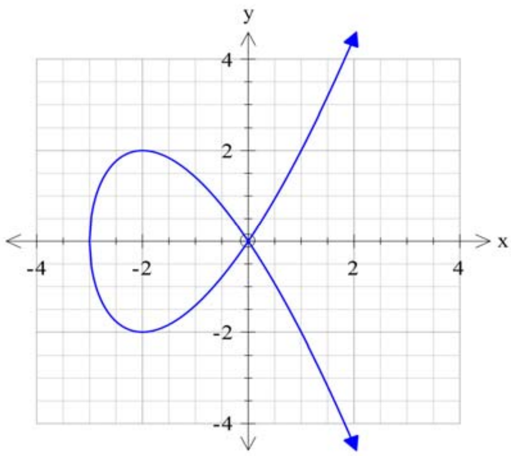
$$k = 0.305$$

$$0.05 = e^{-0.305t}$$

$t = 9.808$ ie. 7PM to the nearest hour

**Question 18** **(9 marks)**

Tschirnhauser's cubic (pictured right) is defined in cartesian form by the equation $y^2 = x^3 + 3x^2$.




(a) Using an appropriate substitution, show algebraically that the area of the loop equals $\frac{24\sqrt{3}}{5}$ units². (6 marks)

$y^2 = x^3 + 3x^2$ $\quad y = \sqrt{x^3+3x^2}$ (top section)

$\int_{-3}^{0} \sqrt{x^3+3x^2}\, dx$

$= \int_{-3}^{0} x\sqrt{x+3}\, dx$

let $u = x+3$ $\quad \frac{du}{dx} = 1$

$x=0 \quad u=3$

$x=-3 \quad u=0$

$= \int_{0}^{3} (u-3)\sqrt{u}\, du$

$= \int_{0}^{3} u^{\frac{3}{2}} - 3u^{\frac{1}{2}}\, du = \left[\frac{2u^{\frac{5}{2}}}{5} - 2u^{\frac{3}{2}}\right]_0^3$

$= \frac{24\sqrt{3}}{5}$

(b) A solid is formed by rotating Tschirnhauser's cubic for $x<0$ about the $y$-axis. Find an expression for determining the volume of this solid and evaluate to 2 d.p. . (3 marks)

$2\times \int_{-3}^{0} 2\pi x y\, dx$

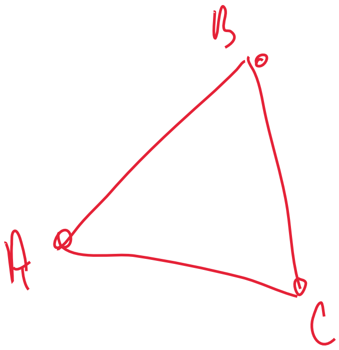
$= 2\times \int_{-3}^{0} 2\pi\, x\sqrt{x^3+3x^2}\, dx$

$= 89.55$ units³

**Question 19** **(5 marks)**

A triangle has vertices $A(k,0,-k)$, $B(k+1,-k,-k)$ and $C(k,-k,0)$.

Determine the value(s) for $k$ given the area of the triangle is $\sqrt{6}$.




$$\vec{AB} = \begin{pmatrix} k+1 \\ -k \\ -k \end{pmatrix} - \begin{pmatrix} k \\ 0 \\ -k \end{pmatrix} = \begin{pmatrix} 1 \\ -k \\ 0 \end{pmatrix}$$

$$\vec{AC} = \begin{pmatrix} k \\ -k \\ 0 \end{pmatrix} - \begin{pmatrix} k \\ 0 \\ -k \end{pmatrix} = \begin{pmatrix} 0 \\ -k \\ k \end{pmatrix}$$

$$\text{Area} = \frac{1}{2}\left|\vec{AB} \times \vec{AC}\right|$$

$$= \frac{1}{2}\left|\begin{pmatrix} 1 \\ -k \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ -k \\ k \end{pmatrix}\right|$$

$$= \frac{1}{2}\begin{vmatrix} i & j & k \\ 1 & -k & 0 \\ 0 & -k & k \end{vmatrix}$$

$$= \frac{1}{2}\left|\begin{pmatrix} k^2 \\ -k \\ -k \end{pmatrix}\right|$$

$$= \frac{1}{2}\sqrt{k^4 + k^2 + k^2}$$

$$= \frac{1}{2}\sqrt{k^2(k^2+2)}$$

$$2\sqrt{6} = k\sqrt{k^2+2}$$

$$4 \times 6 = k^2(k^2+2)$$

$$0 = k^4 + 2k^2 - 24$$

$$0 = (k^2+6)(k^2-4)$$

$$k = \pm 2$$

**END OF SECTION 2**

**Additional working space**

Question number: _________

**Additional working space**

Question number: _________

**Additional working space**

Question number: _________